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continued...
Let us choose one of basis of $\mathbb{R}$ having the elements: $\alpha_1 = (1,0,0), \alpha_2 = (0,1,0), \alpha_3 = (0,0,1)$. From which one can easily get: $T(1,0,0) = (1,2,1) = T(\alpha_1)$, $T(0,1,0) = (1,1,2) = T(\alpha_2)$, $T(0,0,1) = (1,2,1) = T(\alpha_3)$. Clearly, the set $\{T(\alpha_1), T(\alpha_2), T(\alpha_3)\}$ is linearly dependent. Then, we consider the set $\{T(\alpha_1), T(\alpha_2)\} = \{(1,2,1), (1,1,2)\}$, which is linearly independent. The image set $\text{Im}(T)$ is generated/spanned by $\{T(\alpha_1), T(\alpha_2)\}$, i.e., $\text{Im}(T) = \{T(\alpha_1), T(\alpha_2)\}$. Also, $\dim(\text{Im}(T)) = 2$.
From here, we observe that: $\dim(\ker(T)) + \dim(\text{Im}(T)) = 1 + 2 = 3 = \dim(\mathbb{R}^3) \implies \text{null}(T) + \text{rank}(T) = \dim(V)$.

Theorem 1. *If $T : V \to W$ is a linear transformation, then null$(T) + \text{rank}(T) = \dim(V)$. This theorem is known as Rank-Nullity Theorem.*

Theorem 2. *Let V and W be two vector spaces over a field $\mathbb{F}$, and $T : V \to W$ be a linear transformation. Let $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a basis of V, then $\{T(\alpha_1), T(\alpha_2), \ldots, T(\alpha_n)\}$ generates* $\text{Im}(T)$.

Example 2. Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be such that $T(x_1, x_2, x_3) = (x_2 + x_3, x_3 + x_1, x_1 + x_2, x_1 + x_2 + x_3)$.

(i) Verify T is linear.

(ii) Find $\ker(T)$.

(iii) Verify $\{T(1,0,0), T(0,1,0), T(0,0,1)\}$ is linearly independent set.

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Solution. (i) Let us consider $\alpha = (x_1, x_2, x_3)$ and $\beta = (y_1, y_2, y_3)$. So,

$$\begin{aligned} T(\alpha + \beta) &= T(x_1 + y_1, x_2 + y_2, x_3 + y_3) \\ &= (x_2 + y_2 + x_3 + y_3, x_3 + y_3 + x_1 + y_1, x_1 + y_1 + x_2 + y_2, x_1 + y_1 + x_2 + y_2 + x_3 + y_3) \\ &= (x_2 + x_3 + y_2 + y_3, x_3 + x_1 + y_3 + y_1, x_1 + x_2 + y_1 + y_2, x_1 + x_2 + x_3 + y_1 + y_2 + y_3) \\ &= T(x_1, x_2, x_3) + T(y_1, y_2, y_3) = T(\alpha) + T(\beta) \end{aligned}$$

Similarly, one can check that $T(c\alpha) = cT(\alpha)$. $\implies T$ is linear.
(ii) By definition, $\ker(T) = \left\{\alpha = (x_1, x_2, x_3) \in \mathbb{R}^3 : T(\alpha) = \vec{0} \in \mathbb{R}^4\right\}$. This leads to a system of equations: $x_2 + x_3 = 0$, $x_3 + x_1 = 0$, $x_1 + x_2 = 0$, $x_1 + x_2 + x_3 = 0$, which has the solution: $x_1 = 0, x_2 = 0, x_3 = 0$. Therefore, $\ker(T) = \{(0,0,0)\}$ $\{\theta\}$.
(iii) Now let $\alpha_1 = (1,0,0), \alpha_2 = (0,1,0), \alpha_3 = (0,0,1)$. Then, $T(1,0,0) = (0,1,1,1), T(0,1,0) = (1,0,1,1), T(0,0,1) = (1,1,0,1)$. Let $c_1, c_2, c_3 \in \mathbb{F}$ be such that

$$\begin{aligned} & c_1T(\alpha_1) + c_2T(\alpha_2) + c_3T(\alpha_3) = \vec{0} \\ \implies & c_1(0,1,1,1) + c_2(1,0,1,1) + c_3(1,1,0,1) = (0,0,0,0) \end{aligned}$$

leading to: $c_2 + c_3 = 0$, $c_1 + c_3 = 0$, $c_1 + c_2 = 0$, $c_1 + c_2 + c_3 = 0$. So, $c_1 = 0, c_2 = 0, c_3 = 0$. Clearly, all c_i's, $i = 1, 2, 3$ are zero. The set $\{T(\alpha_1), T(\alpha_2), T(\alpha_3)\}$ is linearly independent.

Example 3. Let us consider a map $T : \mathbb{R}^3 \to \mathbb{R}^2$ be such that $T(x_1, x_2, x_3) = (x_1 + x_2, x_3 + x_2)$.

(i) Verify if T is linear.

(ii) If yes, find $\ker(T)$ and $\mathrm{Im}(T)$.

Example 4. Let $T : \mathbb{R}^7 \to \mathbb{R}^8$ be such that $T(x_1, x_2, \ldots, x_7) = (x_1+x_2, x_2+x_3, \ldots, x_7+x_8, x_8+x_1)$.

(i) Verify if T is linear.

(ii) If yes, find $\ker(T)$ and $\mathrm{Im}(T)$.

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1 Matrix Representation of a Linear Map

Let V and W be two vector spaces, and $T : V \to W$ is a linear map. Let $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ be the ordered basis in V and $(\beta_1, \beta_2, \ldots, \beta_n)$ be the ordered basis in W. Then, each $T(\alpha_i)$ can be written as the linear combination of β_i's, $\forall i$, i.e.,

$$\begin{aligned} y_1 = T(\alpha_1) &= a_{11}\beta_1 + a_{21}\beta_2 + \ldots + a_{m1}\beta_m \\ y_2 = T(\alpha_2) &= a_{12}\beta_1 + a_{22}\beta_2 + \ldots + a_{m2}\beta_m \\ &\vdots \\ y_n = T(\alpha_n) &= a_{1n}\beta_1 + a_{2n}\beta_2 + \ldots + a_{mn}\beta_m \end{aligned}$$

where a_{ij}, $\forall i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$ can be uniquely determined. then, the corresponding matrix A is given by

$$A_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

We see that $y = A^t B$, where $B = (\beta_1, \beta_2, \ldots, \beta_m)$.

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Example 1. A linear map $T : \mathbb{R}^3 \to \mathbb{R}^2$ is given by $T(x_1, x_2, x_3) = (3x_1 - 2x_2 + x_3, x_1 - 3x_2 - 2x_3)$. Find the matrix representation of T with respect to the ordered/standard basis in $\mathbb{R}^3$.

Solution. Lets us take the ordered basis in $\mathbb{R}^3$ as $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$. We now write:

$$\begin{aligned} T(\alpha_1) = T(1, 0, 0) &= (3, 1) \\ T(\alpha_2) = T(0, 1, 0) &= (-2, -3) \\ T(\alpha_3) = T(0, 0, 1) &= (1, -2) \end{aligned}$$

The corresponding matrix representation of T will be: $A = \begin{bmatrix} 3 & -2 & 1 \\ 1 & -3 & -2 \end{bmatrix}_{2\times 3}$.

Example 2. For the same linear transformation as the previous example, find the matrix representation of T with respect to the ordered basis $\{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$ of $\mathbb{R}^3$ and $\{(1, 0), (0, 1)\}$ of $\mathbb{R}^2$.

Solution. We have: $T(1, 1, 0) = (1, -2)$, $T(1, 0, 1) = (4, -1), T(0, 1, 1) = (-1, -5)$. So, we can write:

$$\begin{aligned} T(1, 1, 0) = (1, -2) &= 1(1, 0) - 2(0, 1) \\ T(1, 0, 1) = (4, -1) &= 4(1, 0) - 1(0, 1) \\ T(0, 1, 1) = (-1, -5) &= -1(1, 0) - 5(0, 1) \end{aligned}$$

and hence, the matrix representation will be $A = \begin{bmatrix} 1 & 4 & -1 \\ -2 & -1 & -5 \end{bmatrix}_{2\times 3}$.

Example 3. Let the transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $T(x_1, x_2, x_3) = (2x_1 + x_2 - x_3, x_2 + 4x_3, x_1 - x_2 + 3x_3)$. Find the matrix representation of T with respect to the ordered basis in $\mathbb{R}^3$.

Solution. Choose the basis: $S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ and proceed as previous examples.

Example 4. The matrix representation of a linear map $T : \mathbb{R}^3 \to \mathbb{R}^3$ with respect to the ordered basis $\{(0, 1, 1), (1, 0, 1), (1, 1, 0)\}$ of $\mathbb{R}^3$ is given by $A = \begin{bmatrix} 0 & 3 & 0 \\ 2 & 3 & -2 \\ 2 & -1 & 2 \end{bmatrix}$. Find the linear transformation of A.

Solution. We observe that:

$$\begin{aligned} T(0,1,1) &= 0(0,1,1) + 2(1,0,1) + 2(1,1,0) = (4,2,2) \\ T(1,0,1) &= 3(0,1,1) + 3(1,0,1) - (1,1,0) = (2,2,6) \\ T(1,1,0) &= 0(0,1,1) - 2(1,0,1) + 2(1,1,0) = (0,2,-2) \end{aligned}$$

Let (x, y, z) be in $\mathbb{R}^3$ such that $(x, y, z) = c_1(0, 1, 1) + c_2(1, 0, 1) + c_3(1, 1, 0)$. Which gives: $c_1 = \frac{y+z-x}{2}$, $c_2 = \frac{x+z-y}{2}$, $c_3 = \frac{x+y-z}{2}$.
Therefore,

$$\begin{aligned} (x, y, z) &= \frac{y+z-x}{2}(0,1,1) + \frac{x+z-y}{2}(1,0,1) + \frac{x+y-z}{2}(1,1,0) \\ T(x, y, z) &= \frac{y+z-x}{2}T(0,1,1) + \frac{x+z-y}{2}T(1,0,1) + \frac{x+y-z}{2}T(1,1,0) \\ &= \frac{y+z-x}{2}(4,2,2) + \frac{x+z-y}{2}(2,2,6) + \frac{x+y-z}{2}(0,2,-2) \\ &= (-x+y+3z, x+y+z, x-3y+5z) \end{aligned}$$

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which certainly is the required linear map.

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1 Eigenvalues and Eigenvectors

Let A be an $n \times n$ matrix and $\lambda \in \mathbb{F}$. Then, $\det(A - \lambda I)$ is said to be a Characteristic Polynomial of A where $I_{n\times n}$ is the identity matrix, and is denoted by $\Psi_\lambda(A)$. Furthermore, $\Psi_\lambda(A) = 0$ is called the Characteristic Equation. The roots of $\Psi_\lambda(A) = 0$ are called the Eigenvalues of the matrix A. If a root of $\Psi_\lambda(A) = 0$ has multiplicity r, then it is said to be an r-fold eigenvalue. For an r-fold eigenvalue λ, r is called the Algebraic Multiplicity of λ, and the rank of characteristic subspace corresponding to λ is called the Geometric Multiplicity.

Example 1. Find the eigenvalue of the matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Solution. Let $\lambda \in \mathbb{F}$. Then, the characteristic polynomial is, $\det(A - \lambda I) = \begin{vmatrix} 0-\lambda & 1 \\ 1 & 0-\lambda \end{vmatrix} = \lambda^2 - 1$. Therefore, the characteristic equation is given by $\det(A - \lambda I) = 0 \implies \lambda^2 - 1 = 0 \implies \lambda = -1, 1$. The required eigenvalues are -1 and 1.

Example 2. Find the eigenvalues of $A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$.

Solution. Let $\lambda \in \mathbb{F}$. Then, the characteristic polynomial is given by

$$\Psi_\lambda(A) = \det(A - \lambda I) = \det\left\{ \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\} = \begin{vmatrix} 1-\lambda & 1 & 1 \\ -1 & -1-\lambda & -1 \\ 0 & 0 & 1-\lambda \end{vmatrix}.$$

The characteristic equation is $\Psi_\lambda(A) = 0 \implies \begin{vmatrix} 1-\lambda & 1 & 1 \\ -1 & -1-\lambda & -1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$. After solving the determinant, we get $(1-\lambda)\lambda^2 = 0 \implies \lambda = 0, 0, 1$. Clearly, 0 is an eigenvalue with the algebraic multiplicity 2 and 1 has the algebraic multiplicity 1.

Now let $\lambda = \lambda_1$ be an eigenvalue of $A_{n\times n}$. Then, there exists a non-zero vector $X \in \mathbb{R}^n$ such that $AX = \lambda_1 X$.

Example 3. Find the eigenvectors corresponding to the eigenvalues we found in the previous example.

Solution. We are given $A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$. We know that $\lambda = 0, 0, 1$. Let $X = (x, y, z) \in \mathbb{R}^3$

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be the eigenvector corresponding to $\lambda = 0$. Then,

$$AX = \lambda X$$
$$\implies \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$
$$\implies \begin{bmatrix} x+y+z \\ -x-y-z \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We get: $z = 0$, $x + y = 0 \implies x = -y = -k$, where $k \in \mathbb{R}$. So, the required eigenvector corresponding to the eigenvalue 0 is: $X = k(-1, 1, 0)$, $k \in \mathbb{R}$.
Now let $\lambda = 1$. Then, the eigenvector can be obtain

$$AX = \lambda X$$
$$\implies \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 1 \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$
$$\implies \begin{bmatrix} x+y+z \\ -x-y-z \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

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We get: $y + z = 0$, $x + y = -(y + z) = 0$. So, we get $x = z = k \in \mathbb{R}$. The required eigenvector corresponding to the eigenvalue 1 is: $X = k(1, -1, 1)$, $k \in \mathbb{R}$.

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Example 4. Consider $A = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}$. Find the characteristic equation.

Solution. Let $\lambda \in \mathbb{F}$. Then, the characteristic equation is $\det(A - \lambda I_2) = 0 \implies \begin{vmatrix} 2-\lambda & 1 \\ 3 & 5-\lambda \end{vmatrix} = 0 \implies \lambda^2 - 7\lambda + 7 = 0$.

Cayley-Hamilton Theorem. *Every square matrix A of order n satisfies its own characteristic equation.*

Example 5. Verify the Cayley-Hamilton theorem for the matrix $A = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}$.

Solution. The characteristic equation for A is given by $\lambda^2 - 7\lambda + 7 = 0$ as we found out previously.
Goal: $A^2 - 7A + 7I = \vec{0}$

$$\begin{aligned}
\text{L.H.S.} &\implies A \cdot A - 7A + 7I \\
&\implies \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix} - 7\begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&\implies \begin{bmatrix} 7 & 7 \\ 14 & 28 \end{bmatrix} - \begin{bmatrix} 14 & 7 \\ 21 & 35 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} \\
&\implies \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \implies \mathbf{0}_{2\times 2} \implies \text{R.H.S.}
\end{aligned}$$

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Example 6. Using the Cayley-Hamilton theorem, find the inverse of $\begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}$.

Solution. We know $\begin{vmatrix} 2 & 1 \\ 3 & 5 \end{vmatrix} = 7 > 0$. Again,

$$\begin{aligned}
& A^2 - 7A + 7I = 0 \\
\implies & A(A - 7I) = -7I \\
\implies & A^{-1} = -\frac{1}{7}(A - 7I) \\
\implies & A^{-1} = \frac{1}{7}\left(\begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}\right) \\
\implies & A^{-1} = \frac{1}{7}\begin{bmatrix} 5 & -1 \\ -3 & 2 \end{bmatrix}.
\end{aligned}$$

Proposition 1. *If A is a singular matrix, then 0 is an eigenvalue of A.*

Proposition 2. *The eigenvalues in a diagonal matrix are its diagonal entries.*

Proposition 3. *If λ is an eigenvalue of a non-singular matrix, then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.*

Eigenvector. Let A be an $n \times n$ matrix over a field $\mathbb{F}$. A non-null vector X belonging to $V_n(k)$ is said to be an eigenvector of A, if there exists a scalar $\lambda \in \mathbb{F}$ such that $AX = \lambda X$ holds.

Example 7. Let $A = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix}$. Find the eigenvalues and eigenvectors of A.

Solution. Let $\lambda \in \mathbb{F}$. Then, $\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 3 \\ 4 & 5-\lambda \end{vmatrix} = 0 \implies \lambda = -1, 7$.

Let $X \in \mathbb{R}^2$ be the eigenvector corresponding to the eigenvalue -1, then

$$\begin{aligned} & AX = \lambda X \\ \implies & \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -1 \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ \implies & \begin{bmatrix} x_1 + 3x_2 \\ 4x_1 + 5x_2 \end{bmatrix} = - \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ \implies & \begin{bmatrix} 2x_1 + 3x_2 \\ 4x_1 + 6x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

We get $x_1 = -\frac{3}{2}c$, where $x_2 = c \in \mathbb{R}$. Then, the required eigenvector is $X = \begin{bmatrix} -\frac{3}{2} \\ 1 \end{bmatrix} c \ ; \ c \in \mathbb{F}$.

For $\lambda = 7$, we have

$$\begin{aligned} & AX = \lambda X \\ \implies & \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 7 \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ \implies & \begin{bmatrix} x_1 + 3x_2 \\ 4x_1 + 5x_2 \end{bmatrix} = \begin{bmatrix} 7x_1 \\ 7x_2 \end{bmatrix} \\ \implies & \begin{bmatrix} -6x_1 + 3x_2 \\ 4x_1 - 2x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

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We get $x_1 = \frac{1}{2}c$, where $x_2 = c \in \mathbb{R}$. Then, the required eigenvector is $X = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} c \ ; \ c \in \mathbb{F}$.